**AP Calculus BC**
**Review — Techniques of Integration (Chapter 8), Part 1**

## Things to Know and Be Able to Do

- Perform integration by parts: $\int u\,dv = uv - \int v\,du$
- Evaluate integrals of products of trigonometric functions using Pythagorean identities and double- and half-angle formulas
- Evaluate integrals of functions involving radicals by using trigonometric substitutions and identities:
    - Given an integrand of the form $\sqrt{a^2 - x^2}$, substitute $x = a\sin\theta$ and use $\cos^2\theta = 1 - \sin^2\theta$
    - Given an integrand of the form $\sqrt{a^2 + x^2}$, substitute $x = a\tan\theta$ and use $\sec^2\theta = 1 + \tan^2\theta$
    - Given an integrand of the form $\sqrt{x^2 - a^2}$, substitute $x = a\sec\theta$ and use $\tan^2\theta = \sec^2\theta - 1$
- Evaluate integrals of rational functions by using partial fraction decomposition, synthetic division, and polynomial long division
- Know when each of the above strategies is useful and know when to combine or repeat them
- Be able to evaluate both definite and indefinite integrals by all of these methods

## Practice Problems

*These problems should be done without the use of a calculator.*

**1–6** Evaluate each integral. Show all of your work, substitutions, etc.

**1** $\int \tan^{-1} x\,dx$

**2** $\int_0^1 \frac{x}{\left(1+x^2\right)^2}\,dx$

**3** $\int \sec^4 x \tan^3 x\,dx$

**4** $\int \frac{4}{x^2 - 2x}\,dx$

**5** $\int \frac{dx}{\left(4-x^2\right)^{3/2}}$

**6** $\int \frac{x^3+4x+1}{x^2+1}\,dx$

**7** Let $R$ be the region in the first quadrant bounded between the graphs of $y = e^x$, the $y$-axis, the $x$-axis, and the line $x = 2$. Find the volume of the solid generated when $R$ is revolved about the $y$-axis. Show all of your work.

**8a** Find the partial fraction decomposition of the function $f(x) = \frac{x+1}{x^3 - x^2}$.

**8b** Use your answer from part **a** to evaluate $\int_2^3 \frac{x+1}{x^3 - x^2}\,dx$.

**9** $\int x\cos x\,dx =$

**a** $x\sin x - \cos x + C$ **b** $x\sin x + \cos x + C$ **c** $-x\sin x + \cos x + C$
**d** $x\sin x + C$ **e** $\frac{1}{2}x^2 \sin x + C$

**10** $\int_0^3 \sqrt{9 - x^2}\,dx =$

**a** 9 **b** 27 **c** $\frac{9\pi}{4}$ **d** $\frac{9\pi}{2}$ **e** $9\pi$

**11** $\int_0^{\pi/4} \sin^2 t\,dt =$

**a** $\frac{1}{2}$ **b** $\frac{\pi}{8}$ **c** $\frac{\pi}{8} + \frac{1}{4}$ **d** $\frac{\pi}{8} + \frac{1}{2}$ **e** $\frac{\pi}{8} - \frac{1}{4}$

**12** $\int \frac{y-1}{y+1}dy =$

**a** $y-2\ln|y+1|+C$ **b** $1-\frac{2}{y+1}+C$ **c** $\ln\left(\frac{|y|}{(y+1)^2}\right)+C$

**d** $1-2\ln|y+1|+C$ **e** $\ln\left|\frac{e^y}{y+1}\right|+C$

**13** $\int \frac{dx}{(x-1)(x+2)} =$

**a** $\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$ **b** $\frac{1}{3}\ln\left|\frac{x+2}{x-1}\right|+C$ **c** $\frac{1}{3}\ln|(x-1)(x+2)|+C$

**d** $(\ln|x-1|)(\ln|x+2|)+C$ **e** $\ln\left|(x-1)(x+2)^2\right|+C$

**14** $\int \frac{x}{1+4x^2}dx =$

**a** $\frac{1}{8}\ln(1+4x^2)+C$ **b** $\frac{1}{8(1+4x^2)^2}+C$ **c** $\frac{1}{4}\sqrt{1+4x^2}+C$

**d** $\frac{1}{2}\ln(1+4x^2)+C$ **e** $\frac{1}{2}\tan^{-1}(2x)+C$

## Answers

**1** $x\tan^{-1}x-\frac{1}{2}\ln\left|1+x^2\right|+C$

**2** $\frac{1}{4}$, **3** $\frac{1}{6}\tan^6 x+\frac{1}{4}\tan^4 x+C$

**4** $2\ln\frac{x-2}{2}+C$

**5** $\frac{x}{4\sqrt{4-x^2}}+C$

**6** $\frac{1}{2}x^2+\frac{3}{2}\ln\left|1+x^2\right|+\tan^{-1}x+C$

**7** $2\pi\left(e^2+1\right)$

**8a** $-\frac{2}{x}-\frac{1}{x^2}+\frac{2}{x-1}$

**8b** $2\ln\frac{4}{3}-\frac{1}{6}$

**9** b
**10** c
**11** e

**12** a
**13** a
**14** a

## Solutions

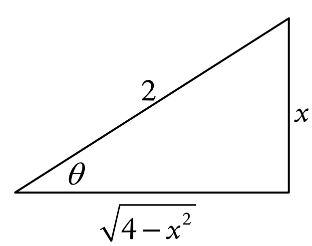
**1** Integrate by parts, using the values $u=\tan^{-1}x$ and $dv=dx$. Then $du=\frac{dx}{1+x^2}$ and $v=x$. Therefore $\int\tan^{-1}x\,dx=x\tan^{-1}x-\int\frac{x}{1+x^2}dx$. The integral that remains can be evaluated by making the substitution $u=1+x^2$, so $du=2x\,dx$ and the integral is $\int\frac{du}{2u}=\frac{1}{2}\ln|u|+C$, or $\frac{1}{2}\ln\left|1+x^2\right|+C$. Therefore the original indefinite integral evaluates to $x\tan^{-1}x-\frac{1}{2}\ln\left|1+x^2\right|+C$.

**2** Make the substitution $u=1+x^2$, so $du=2x\,dx$. The limits then change: $x=0\rightarrow u=1$ and $x=1\rightarrow u=2$. The integral itself is $\int_1^2\frac{1}{2}u^{-2}\,du=-\frac{1}{2}u^{-1}\Big]_1^2=-\frac{1}{4}-\left(-\frac{1}{2}\right)=\frac{1}{4}$.

**3** We want to have one extra factor of $\sec^2 x$, so we change $\sec^4 x$ to $\sec^2 x\left(\tan^2 x+1\right)$. The integral therefore becomes $\int\sec^2 x\left(\tan^2 x+1\right)\tan^3 x\,dx$, which can be rearranged to $\int\sec^2 x\left(\tan^5 x+\tan^3 x\right)dx$. We then make the substitution $u=\tan x$, so $du=\sec^2 x\,dx$. Now we have $\int\left(u^5+u^3\right)du=\frac{1}{6}u^6+\frac{1}{4}u^4+C$, or $\frac{1}{6}\tan^6 x+\frac{1}{4}\tan^4 x+C$.

**4** The integrand is a rational function, so we use the method of partial fractions. The denominator factors to $x(x-2)$, so the partial fraction decomposition takes the form $\frac{A}{x}+\frac{B}{x-2}=\frac{4}{x(x-2)}$. This lets us determine $(A+B)x-2A=4$, meaning $A+B=0$ and $-2A=4$. This system can be solved for $A=-2$ and $B=2$, so the integral is $\int\left(-2x^{-1}+2(x-2)^{-1}\right)dx$. This is $-2\ln|x|+2\ln|x-2|+C$, which can also be written as $2\ln\frac{x-2}{2}+C$.

**5** Consider the triangle shown at right. Make the substitution $x=2\sin\theta$, giving $dx=2\cos\theta\,d\theta$. Now the integral is $\int\frac{2\cos\theta\,d\theta}{\left(4\cos^2\theta\right)^{3/2}}=\int\frac{2\cos\theta\,d\theta}{8\cos^3\theta}=\int\frac{1}{4}\sec^2\theta\,d\theta$, which is $\frac{1}{4}\tan\theta+C$. Looking at the triangle, this is $\frac{1}{4}\left(\frac{x}{\sqrt{4-x^2}}\right)+C=\frac{x}{4\sqrt{4-x^2}}+C$.




**6** Again, this is a rational function. To find the partial fraction decomposition, carry out polynomial long division:

$$\begin{array}{r|l}
 & x \\
x^2+1 & x^3+0x^2+4x+1 \\
 & -\left(x^3 \qquad\quad +x\right) \\
\hline
 & \quad 0 \;\; +0 \quad\;\; +3x+1
\end{array}$$

Therefore the integrand may be written as $x+\frac{3x+1}{x^2+1}$. We split up the fraction, making our problem fairly simple: $\int\left(x+\frac{3x}{x^2+1}+\frac{1}{x^2+1}\right)dx$. This evaluates to $\frac{1}{2}x^2+\frac{3}{2}\ln\left|x^2+1\right|+\tan^{-1}x+C$.

**7** We use cylindrical shells centered around the $y$-axis. Each shell has radius $x$, thickness $dx$, and height $y=e^x$. Therefore each shell has volume $dV=2\pi xe^x dx$, for a total volume of $V=\int_0^2 2\pi xe^x dx=2\pi\int_0^2 xe^x dx$. We evaluate this integral by parts, using $u=x$ and $dv=e^x dx$. Therefore the integral is $2\pi\left(xe^x\Big]_0^2-\int_0^2 e^x dx\right)$, which is equivalent to $2\pi\left(\left(2e^2-0\right)-e^x\Big]_0^2\right)=2\pi\left(2e^2-\left(e^2-1\right)\right)=2\pi\left(e^2+1\right)$.

**8a** The denominator can be factored to $x^2(x-1)$, so the partial fraction decomposition takes the form $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}=\frac{x+1}{x^2(x-1)}$. This can be rearranged to $x+1=x^2(A+C)+x(-A+B)-B$, so we have the system $\begin{cases}A+C=0\\-A+B=1\\-B=1\end{cases}$ meaning $(A,B,C)=(-2,-1,2)$. Therefore the decomposition is $-\frac{2}{x}-\frac{1}{x^2}+\frac{2}{x-1}$.

**8b** Using the above equivalent to the integrand lets us determine that an antiderivative is $-2\ln|x|+\frac{1}{x}+2\ln|x-1|$, so we evaluate that from 2 to 3. This gives $-2\ln 3+\tfrac{1}{3}+2\ln 2-\left(-2\ln 2+\tfrac{1}{2}+2\ln 1\right)$, which is equivalent to $2\ln\tfrac{4}{3}-\tfrac{1}{6}$.

**9** Integrate by parts, using $u=x$ and $dv=\cos x\,dx$. This gives $x\sin x-\int\sin x\,dx$, which is $x\sin x+\cos x+C$, or **b**.

**10** The integral gives one quarter of a circle of radius 3. That area is $A=\tfrac{1}{4}\left(3^2\pi\right)=\tfrac{9}{4}\pi$, choice **c**.

**11** We can use the half-angle identity $\tfrac{1}{2}(1-\cos 2t)=\sin^2 t$. The integral is then $\int_0^{\pi/4}\tfrac{1}{2}(1-\cos 2t)\,dt=\tfrac{1}{2}t-\tfrac{1}{4}\sin 2t\Big]_0^{\pi/4}$ $=\tfrac{1}{2}\left(\tfrac{\pi}{4}\right)-\tfrac{1}{4}\sin\tfrac{\pi}{2}=\tfrac{\pi}{8}-\tfrac{1}{4}$, or **e**.

**12** Find the partial fraction decomposition of this rational function by polynomial long division:

$$\begin{array}{r} 1 \\ y+1\,\overline{)\,y-1} \\ -(y+1) \\ \hline -2 \end{array}$$

So the integrand may be written as $1-\frac{2}{y+1}$; we want to find $\int\left(1-\frac{2}{y+1}\right)dy$. This is $y-2\ln|y+1|+C$, choice **a**.

**13** Because the denominator of this rational function is already factored, finding its partial fraction decomposition requires one fewer step than usual. We know that it takes the form $\frac{1}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$, which can be rearranged to $x(A+B)+2A-B=1$. Therefore $\begin{cases}A+B=0\\2A-B=1\end{cases}$ so $(A,B)=\left(\tfrac{1}{3},-\tfrac{1}{3}\right)$ and the integral can be written as $\int\left(\frac{1/3}{x-1}-\frac{1/3}{x+2}\right)dx=\tfrac{1}{3}\ln|x-1|-\tfrac{1}{3}\ln|x+2|+C$. We can write this as $\tfrac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$, choice **a**.

**14** Make the substitution $u=1+4x^2$, so $du=8x\,dx$. The integral is now $\int\tfrac{1}{8}\left(\frac{du}{u}\right)=\tfrac{1}{8}\ln|u|+C$, so reversing the substitution gives $\tfrac{1}{8}\ln\left|1+4x^2\right|+C$, choice **a**.