MV Calculus
Chapter 17
Practice

1. Integrate $f(x,y,z)=\sqrt{x^2+z^2}$ over the circle $\mathbf{r}(t)=(a\cos t)\hat{\mathbf{j}}+(a\sin t)\hat{\mathbf{k}}$ for $0\le t\le 2\pi.$

Find $d\mathbf{r}=\langle 0,-a\sin t,a\cos t\rangle dt$ so $\|d\mathbf{r}\|=a\,dt.$ Then the integral is

$$\int_0^{2\pi}\sqrt{0^2+(a\sin t)^2}\,\|d\mathbf{r}\|=\int_0^{2\pi}\sqrt{a^2\sin^2 t}\,a\,dt=\int_0^{2\pi}a^2|\sin t|\,dt=4a^2$$

2. Evaluate the integral $\int_{(-1,1,1)}^{(4,-3,0)}\frac{dx+dy+dz}{\sqrt{x+y+z}}.$

This is $2\sqrt{x+y+z}\Big]_{(-1,1,1)}^{(4,-3,0)}=2-2=0.$

3. Find the area of the region cut from the plane $x+y+z=1$ by the cylinder $x^2+y^2=1.$

Parameterize the plane as $\begin{cases}x=u\\ y=v\\ z=1-u-v\end{cases}$ so $\mathbf{r}(u,v)=\langle u,v,1-u-v\rangle.$ Then $\mathbf{r}_u=\langle 1,0,-1\rangle$ and $\mathbf{r}_v=\langle 0,1,-1\rangle.$ Thus $\|\mathbf{r}_u\times\mathbf{r}_v\|=\|\langle 1,1,1\rangle\|=\sqrt{3}.$ We then use cylindrical coordinates:

$$\int_0^{2\pi}\int_0^1\sqrt{3}r\,dr\,d\theta=\pi\sqrt{3}.$$

4. Find the work done by the field $\mathbf{F}=2xy\hat{\mathbf{i}}+\hat{\mathbf{j}}+x^2\hat{\mathbf{k}}$ in moving along the line segment from $(0,0,0)$ to $(1,1,1).$

Parameterize the path as $\left\{\begin{matrix}x=t\\ y=t\\ z=t\end{matrix}\right\}:t\in[0,1].$ Then $d\mathbf{r}=\langle 1,1,1\rangle dt.$ Therefore the integral is

$$\int_0^1\mathbf{F}\cdot d\mathbf{r}=\int_0^1\left(2t^2+1+t^2\right)dt=\int_0^1\left(3t^2+1\right)dt=2.$$

5. A wire of constant density lies along the curve $\mathbf{r}(t) = \left(e^t \cos t\right)\hat{\mathbf{i}} + \left(e^t \sin t\right)\hat{\mathbf{j}} + e^t\hat{\mathbf{k}}$ for $0 \le t \le \ln 2.$ Find the moment of inertia about the *z*-axis.

$d\mathbf{r} = \left\langle e^t \cos t - e^t \sin t, e^t \cos t + e^t \sin t, e^t \right\rangle dt = e^t dt \left\langle \cos t - \sin t, \cos t + \sin t, 1 \right\rangle,$ so $\|d\mathbf{r}\| = e^t\sqrt{3}\,dt.$ Now since $I_z = \int \left(x^2 + y^2\right) dm,$ the $x^2 + y^2$ changes to $\left(e^t \cos t\right)^2 + \left(e^t \sin t\right)^2 = e^{2t}.$ Therefore we integrate $\int_0^{\ln 2} e^{2t}\sqrt{3}e^t\,dt = \frac{7}{\sqrt{3}}.$

6. Find the counterclockwise circulation and the outward flux caused by the field $\mathbf{F} = (2xy + x)\hat{\mathbf{i}} + (xy - y)\hat{\mathbf{j}}$ over the square bounded by $x = 0,\ \ x = 1,\ y = 0,$ and $y = 1.$

Firstly, find the field's divergence and curl: $\nabla \cdot \mathbf{F} = x + 2y$ and $\nabla \times \mathbf{F} = y - 2x.$ To find the flux, integrate the divergence over the region: $\int_0^1\int_0^1 (x + 2y)\,dy\,dx = \frac{3}{2},$ and to find the circulation, integrate the curl over the region: $\int_0^1\int_0^1 (y - 2x)\,dy\,dx = -\frac{1}{2}.$

7. Evaluate $\oint_C 3y\,dx + 2x\,dy$ where *C* is the boundary of $0 \le x \le \pi$ and $0 \le y \le \sin x.$

This is probably easiest to recognize as an application of Green's theorem in circulation form: $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx.$ In this case, we have $M = 3y$ and $N = 2x.$ Then $\frac{\partial N}{\partial x} = 2$ and $\frac{\partial M}{\partial y} = 3,$ so we have $\int_0^{\pi}\int_0^{\sin x} (2 - 3)\,dy\,dx = -2.$