AP Calculus BC
Review — Applications of Integration (Chapter 6)

Things to Know and Be Able to Do

- Find the area between two curves by integrating with respect to $x$ or $y$
- Find volumes by approximations with cross sections: disks (cylinders), washers, and other shapes
- Find volume by cylindrical shells: (radius $r$, height $h$, and thickness $dr$ gives volume $dV = 2\pi rh\,dr$)
- Find work done using the formula $W = \int F\,dx$, noting that one common instance of a force is weight

Practice Problems

For all problems, show a correct, labeled diagram and a complete setup of the problem in terms of a single variable. Use correct units where applicable. This is designed to be done with a calculator. Remember, when giving approximate answers, to give three decimal places.

1. Let $R$ be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure at right.
   - a) Find the area of $R$.
   - b) Find the volume of the solid generated when $R$ is revolved about the horizontal line $y = 1$.
   - c) The region $R$ is the base of a solid. For this solid, each cross-section perpendicular to the $x$-axis is a rectangle whose height is 5 times the length of its base in region $R$. Find the volume of this solid.

2. A container is in the shape of a regular square pyramid. The height of the pyramid is 6 ft and the sides of the square base are 4 ft long. The tank is full of a liquid with weight density $68 \text{ lb/ft}^3$. Find the work done in pumping the liquid to a point 4 ft above the top of the tank.

3. Let $R$ be the region in the first quadrant bounded by the graph of $y = x - x^3$ and the $x$-axis. Find the volume of the solid generated when $R$ is revolved about the (a) $x$-axis and (b) $y$-axis

4. If the force $F$, in ft $\cdot$ lb, acting on a particle on the $x$-axis is given by $F(x) = \frac{1}{x^2}$, then the work done in moving the particle from $x = 1$ ft to $x = 3$ ft is equal to
   - a) $2 \text{ ft $\cdot$ lb}$
   - b) $\frac{2}{3} \text{ ft $\cdot$ lb}$
   - c) $\frac{26}{27} \text{ ft $\cdot$ lb}$
   - d) $1 \text{ ft $\cdot$ lb}$
   - e) $\frac{3}{2} \text{ ft $\cdot$ lb}$

5. The base of a solid is a circle of radius $a$, and every plane cross-section perpendicular to one specific diameter is a square. The solid has volume
   - a) $\frac{8}{3}a^3$
   - b) $2\pi a^3$
   - c) $4\pi a^3$
   - d) $\frac{16}{3}a^3$
   - e) $\frac{8\pi}{3}a^3$

6. The region whose boundaries are $y = 3x - x^2$ and $y = 0$ is revolved about the $x$-axis. The resulting solid has volume
   - a) $\pi\int_0^3(9x^2 + x^4)\,dx$
   - b) $\pi\int_0^3(3x - x^2)^2\,dx$
   - c) $\pi\int_0^\sqrt{3}(3x - x^2)\,dx$
   - d) $2\pi\int_0^3\sqrt{9 - 4y}\,dy$
   - e) $\pi\int_0^\frac{3}{4}y^2\,dy$
7 The area of the region enclosed by the graphs of $y = x^2$ and $y = x$ is
\[ a \frac{1}{6} \quad b \frac{1}{3} \quad c \frac{1}{2} \quad d \frac{5}{6} \quad e 1 \]

8 When the region enclosed by the graphs of $y = x$ and $y = 4x - x^2$ is revolved about the y-axis, the volume of the solid generated is given by
\[ a \pi \int_0^1 (x^3 - 3x^2) \, dx \quad b \pi \int_0^3 \left( x^3 - (4x - x^2)^2 \right) \, dx \quad c \pi \int_0^3 (3x - x^3)^2 \, dx \]
\[ d 2\pi \int_0^3 (x^3 - 3x^2) \, dx \quad e 2\pi \int_0^3 (3x^2 - x^3) \, dx \]

9 What is the volume of the solid generated by rotating about the x-axis the region enclosed by the graph of $y = \sec x$ and the lines $x = 0$, $y = 0$, and $x = \frac{\pi}{3}$?
\[ a \frac{\pi}{\sqrt{2}} \quad b \pi \quad c \pi \sqrt{3} \quad d \frac{8\pi}{3} \quad e \pi \ln \left( \frac{1}{3} + \sqrt{3} \right) \]

10 If the region in the first quadrant bounded between the y-axis and the graph of $x = 2y(3 - y)^2$ is revolved about the x-axis, the volume of the solid generated is given by
\[ a \int_0^3 \pi (2y(3 - y^2))^2 \, dy \quad b \int_0^8 2\pi x \left( 2x(3 - x^2) \right) \, dx \quad c \int_0^8 2\pi x (3 - \sqrt{x}) \, dx \]
\[ d \int_0^3 4\pi y^2 (3 - y)^2 \, dy \quad e \int_0^8 \pi \left( 2x(3 - x^2) \right) \, dx \]

11 Find the area enclosed by the graphs of $y = x^3 + 2x^2 - 10x - 12$ and $y = x$.
\[ a \frac{343}{12} \quad b \frac{99}{4} \quad c \frac{160}{3} \quad d \frac{937}{12} \quad e \frac{385}{12} \]
Answers

1a 0.443  
1b 1.424  
1c 1.554  
2 18496 ft⋅lb

3a \( \frac{8\pi}{106} \approx 0.239 \)  
3b \( \frac{4\pi}{15} \approx 0.838 \)  
4b  
5d  
6b  
7a  
8e  
9c  
10d

Solutions

1a First we need to find the beginning of the interval over which to integrate, which is the point of intersection of 
\( y = \sqrt{x} \) with \( y = e^{-3x} \). Therefore we solve \( \sqrt{x} = e^{-3x} \); this cannot be solved for \( x \) exactly, but an approximation can be found: 0.239. Since the top function is \( y = \sqrt{x} \), the bottom function is \( y = e^{-3x} \), and the upper limit is 1, we integrate \( \int_{0.239}^{1} (\sqrt{x} - e^{-3x}) dx \). This can be evaluated as \( \frac{1}{3}e^{-3x} + \frac{1}{2}x^{3/2} \) \( \left[0.239 \right]_{0.239}^{1} \) or just plugged into a calculator; the answer is 0.443.

1b We find this object’s volume using disks centered around the line \( y = 1 \). Each disk has inner radius \( 1 - \sqrt{x} \) and outer radius \( 1 - e^{-3x} \), so each one has area \( dA = \pi \left( (1 - e^{-3x})^2 - (1 - \sqrt{x})^2 \right) \) and, with thickness \( dx \), volume \( dV = \pi \left( (1 - e^{-3x})^2 - (1 - \sqrt{x})^2 \right) dx \). To find the total volume, we integrate \( \int_{0.239}^{1} \pi \left( (1 - e^{-3x})^2 - (1 - \sqrt{x})^2 \right) dx \). Don’t bother finding an antiderivative for the integrand; it’s really ugly and you’ll need to approximate the answer anyway. Your calculator will give you the approximation \( V = 1.424 \).

1c Each rectangle has width \( \sqrt{x} - e^{-3x} \) and height \( 5(\sqrt{x} - e^{-3x}) \). They each have area \( dA = 5\left( \sqrt{x} - e^{-3x} \right)^2 \), and if their thickness is \( dx \), each volume is \( dV = 5\left( \sqrt{x} - e^{-3x} \right)^2 dx \). The total volume is given by \( V = \int_{0.239}^{1} 5\left( \sqrt{x} - e^{-3x} \right)^2 dx \). An approximation to this is \( V = 1.554 \).

2 Consider a square horizontal “slab” of liquid at a height \( h \) below the pyramid’s apex and with side length \( x \). A resulting side view of half the pyramid is shown at right. Clearly, the two triangles are similar, so we can set up the proportion \( \frac{x}{2} = \frac{6}{2} \), meaning \( x = \frac{2}{3}h \). Thus a slab located \( h \) below the apex has side length \( \frac{2}{3}b \), area \( dA = \left( \frac{1}{3}b \right)^2 = \frac{4}{9}b^2 \), and if it has thickness \( dh \), volume \( dV = \frac{4}{9}b^2 dh \). This means that each slab’s weight is \( 68\left( \frac{4}{9}b^2 \right) = \frac{272}{9}b^2 \) \( dh \). Each has to be lifted a distance \( b \) to get to the apex and then a further 4 to the desired point, for a total distance of \( b + 4 \). Therefore the work done to lift each slab is \( dW = \frac{272}{9}b^2 (b + 4) \), and the total work is \( \int_{0}^{6} \frac{272}{9}b^2 (b + 4) db = 18496 \) ft⋅lb.

3a A diagram of the region is shown at right. The volume can be found by disks centered around the \( x \)-axis; each disk has radius \( y = x - x^3 \) and thickness \( dx \), for a volume of \( dV = \pi \left( x - x^3 \right)^2 dx \). The object’s total volume is then given by \( V = \int_{0}^{1} \pi \left( x - x^3 \right)^2 dx = \frac{8\pi}{105} \approx 0.239 \).
3b This requires the method of cylindrical shells, which should be centered around the y-axis. Each shell has radius \( x \), thickness \( dx \), and height \( y = x - x^3 \). So each cylinder has volume \( dV = 2\pi x(x - x^3)dx \). The total volume is given by
\[
V = \int_0^1 2\pi x(x - x^3)dx = \frac{4\pi}{15} \approx 0.838.
\]

4 Since \( W = \int F \, dx \) and the particle is moving from \( x = 1 \) ft to \( x = 3 \) ft under a force of \( F = \frac{1}{x^2} \) lb, the total work done is \( \int_1^3 \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_1^3 = \frac{2}{3} \) ft \cdot lb. This is choice b.

5 The circle is given by \( x^2 + y^2 = a^2 \), so \( x^2 = a^2 - y^2 \). Each square has base \( 2x \) and height \( 2x \) for an area of \( (2x)^2 = 4x^2 \). Since the squares are parallel to the x-axis, they have thickness \( dy \), and each “slab” has volume \( dV = 4x^2 \, dy \). Fortunately, since we know \( x^2 = a^2 - y^2 \), we can substitute that in to find \( dV = 4(a^2 - y^2) \, dy \).

Then the total volume is \( V = \int_a^0 4(a^2 - y^2) \, dy = 4 \left[ a^2y - \frac{1}{3}y^3 \right]_a^0 = \frac{8}{3}a^3 - \left( -\frac{8}{3}a^3 \right) = \frac{16}{3}a^3 \), choice d.

6 The region is shown at right; its left boundary is at \( x = 0 \) and its right boundary is at \( x = 3 \). The volume of the solid described is found by disks centered around the x-axis; each disk has radius \( 3x - x^2 \). If the disks have thickness \( dx \), each one’s volume is \( dV = \pi(3x - x^2)^2 \, dx \), so the total volume is given by \( V = \int_0^3 \pi(3x - x^2)^2 \, dx \). This is choice b.

7 The region is shown at left; its right boundary is at \( x = 1 \); the “top” function is \( y = x \), and the “bottom” function is \( y = x^2 \). Therefore, the region’s area is \( A = \int_0^1 (x - x^2) \, dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \bigg|_0^1 = \frac{1}{2}(1)^2 - \frac{1}{3}(1)^3 - \frac{1}{2}(0)^2 - \frac{1}{3}(0)^3 = \frac{1}{6} \), choice a.

8 The region is shown at right; its left boundary is at \( x = 0 \) and its right boundary is at \( x = 3 \). We can find the volume of the solid described with cylindrical shells centered around the y-axis. Each shell has radius \( x \), thickness \( dx \), and height \( (4x - x^2) - x = 3x - x^2 \). Therefore the volume of the solid is \( 2\pi \int_0^3 (3x - x^2) \, dx \), or \( 2\pi \int_0^3 (3x^2 - x^3) \, dx \), choice e.

9 The region is shown at left. We can find the volume of the solid with disks centered around the x-axis. Each disk has radius \( y = \sec x \) and thickness \( dx \), so its volume is \( dV = \pi \sec^2 x \, dx \). The total volume is thus \( \int_0^{\pi/3} \pi \sec^2 x \, dx = \pi \tan x \bigg|_0^{\pi/3} = \pi \sqrt{3} \), choice c.
10. The region is shown at right. Its lower boundary is \( y = 0 \) and its upper boundary is \( y = 3 \). Finding the volume described is tricky; we need to use cylindrical shells centered around the \( x \)-axis. Each shell has radius \( y \), thickness \( dy \), and height \( x = 2y(3 - y)^2 \). The volume of each shell is given by \( dV = 2\pi y(2y(3 - y)^2)dy \), so the total volume is \( dV = \int_0^3 2\pi y(2y(3 - y)^2)dy \). The total volume is \( 4\pi \int_0^3 y^2(3 - y)^2 dy \), choice d.

11. The graphs with the two regions in question shown are at right. The left boundary of the left region is \( x = -4 \), the curves intersect at \( x = -1 \), and the right boundary of the right region is \( x = 3 \). Since in the left region the "top" function is \( y = x^3 + 2x^2 - 10x - 12 \) while in the right region the "top" function is \( y = x \), the left region’s area is \( \int_{-4}^{-1} \left( (x^3 + 2x^2 - 10x - 12) - x \right) dx = \int_{-4}^{-1} (x^3 + 2x^2 - 11x - 12) dx \) and the right’s is \( \int_{-1}^{3} \left( x - (x^3 + 2x^2 - 10x - 12) \right) dx = \int_{-1}^{3} (-x^3 - 2x^2 + 11x + 12) dx \). The first integral is evaluated as \( \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{11}{2}x^2 - 12x \bigg|_{-4}^{-1} = \frac{99}{4} \), and the second as \( -\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{11}{2}x^2 + 12x \bigg|_{-1}^{3} = \frac{160}{3} \). The total area is thus \( \frac{99}{4} + \frac{160}{3} = \frac{937}{12} \), choice d.