Things to Know

- The derivative of a function at a point may be interpreted as the slope of a tangent line to the graph at that point.
- The derivative of a function is itself a function.
- The derivative of a function \( f(x) \) with respect to \( x \) is denoted as \( \frac{df}{dx} \) or \( f'(x) \).
- The derivative of a function \( f(x) \) at \( x = a \) is denoted as \( \frac{df}{dx} \bigg|_{x=a} \) or \( f'(a) \).
- A derivative of a function \( f(x) \) with respect to \( x \) at a point \( x = a \) is defined as \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \); these definitions are equivalent given the definition \( b = x - a \).
- Be able to evaluate limits by recognizing them as derivatives of either of the forms above.
- Not all functions are differentiable; for example, \( f(x) = |x| \) is nondifferentiable at \( x = 0 \) because it is “sharp” at that point. Continuity is necessary but not sufficient for differentiability.
- The Sum Rule: \( \frac{d}{dx} (f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx} \)
- The Constant Multiple Rule: \( \frac{d}{dx} (cf(x)) = c \frac{df}{dx} \)
- The Sum Rule may be combined with the Constant Multiple Rule to form the Difference Rule: \( \frac{d}{dx} (f(x) - g(x)) = \frac{df}{dx} - \frac{dg}{dx} \)
- The Product Rule: \( \frac{d}{dx} (f(x)g(x)) = f(x) \frac{dg}{dx} + g(x) \frac{df}{dx} \)
- The Quotient Rule: \( \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{df}{dx} - f(x) \frac{dg}{dx}}{g(x)^2} \) (convenient mnemonic: “low dee-high minus high dee-low/draw a line and square below”)
- The Power Rule: \( \frac{d}{dx} (x^n) = nx^{n-1} \) and its derivation for positive integers \( n \) with the Binomial Theorem (you needn’t be able to do it, but you should understand it). Know that this is valid for all real \( n \) even though we have only proven it for the integers.
- Important limits: \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \) and \( \lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \)
The derivatives of trigonometric functions (you may find it easier to memorize only the first three and derive the latter three when necessary)

- \( \frac{d}{dx}(\sin x) = \cos x \)
- \( \frac{d}{dx}(\cos x) = -\sin x \)
- \( \frac{d}{dx}(\tan x) = \sec^2 x \)
- \( \frac{d}{dx}(\sec x) = \tan x \sec x \)
- \( \frac{d}{dx}(\csc x) = -\cot x \csc x \)
- \( \frac{d}{dx}(\cot x) = -\csc^2 x \)

Find elementary antiderivatives by recognizing functions as derivatives using any of the following rules or combinations thereof: Power Rule, Product Rule, Quotient Rule. Remember to add a constant.

**Practice Problems**

*Do not use a calculator.*

1 Find each derivative.

   a Find \( f'(x) \) if \( f(x) = 4\cos x - \frac{5}{x} + \pi \)
   
   b Find \( \frac{dp}{dx} \) if \( p = \frac{\cos x}{1+x^2} \)
   
   c Find \( h'(x) \) if \( h(x) = 2x^{\sqrt{6}} + \csc x + \frac{x^4}{6} \)
   
   d Find \( \frac{dy}{dx} \) if \( y = \tan x \csc x \)
   
   e Find \( \frac{dx}{dt} \) if \( x = \frac{-9}{\sqrt{t^3}} + \sec t - \frac{\sqrt{5}}{\pi} \)

2 Suppose that \( y \) and \( z \) are differentiable functions of \( x \), and suppose that some values for \( y(x) \), \( z(x) \), \( y'(x) \), and \( z'(x) \) are given in the table to the right. Find the values of the following derivatives:

   a \( \frac{d}{dx}(yz) \) at \( x = 2 \)
   
   b \( \frac{d}{dx}\left(\frac{z}{y}\right) \) at \( x = 4 \)

3 Evaluate \( \lim_{x \to \pi} \frac{x\cos x + \pi}{x - \pi} \). Show your work.

4 Find a function \( h(x) \) that has the given derivative \( h'(x) \).

   a \( h'(x) = 4x^3 \cot x - x^4 \csc^2 x \)
   
   b \( h'(x) = \frac{3\sqrt{x} \csc x + 2x\sqrt{x} \csc x \cot x}{\csc^2 x} \)

5 The function \( f(x) = x^3 + ax^2 + bx + 5 \) has a horizontal tangent at the point \( (3, -13) \). Show use of calculus to find the constants \( a \) and \( b \).

6 Let \( h(t) \) be the number of hours per week that a student taking Calculus BC spends on homework and studying, where \( t \) is time measured in weeks after September 1. Translate each of the following into a sentence about students and homework hours.

   a \( h(19) = 8 \) and \( h'(19) = 2 \)
   
   b \( h(36) = 1 \) and \( h'(36) = -1 \)
7 Given a graph of $f'(x)$ at right, and $f(3) = -1.$
   a Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 3$.
   b Could the line $4x + y = 8$ be the line tangent to the graph of $y = f(x)$ at $x = -2$? Why or why not?
   c Find the value of $h'(3)$, if $h(x) = \frac{f(x)}{x^2 + 1}$.

8 Given the graph $y = f(x)$ shown, graph the function $y = f'(x)$ on the axes provided.

9 If $f(x) = \frac{x}{\tan x}$, then $f'(\pi/4) =$
   a $\frac{1}{2}$           b $\frac{1}{2}$           c $1 + \frac{\pi}{2}$
   d $\frac{\pi}{2} - 1$   e $1 - \frac{\pi}{2}$

10 An equation of the line tangent to the graph of $y = x - \cos x$ at the point $(0,1)$ is
   a $y = 2x + 1$           b $y = x + 1$           c $y = x$
   d $y = x - 1$           e $y = 0$

11 $\lim_{h \to 0} \frac{(-2 + h)^4 - 16}{h} =$
   a $64$           b $-32$           c $16$
   d $0$           e undefined

12 If $u$, $v$, and $w$ are nonzero differentiable functions, then the derivative of $\frac{uv}{w}$ is
   a $\frac{u'v + uv'}{w'}$           b $\frac{u'v'w - uvw'}{w^2}$
   d $\frac{u'v'w + uvw'}{w^2}$
   e $\frac{uvw' - uv'w - u'vw}{w^2}$

13 What is the instantaneous rate of change at $x = 2$ of $f(x) = \frac{x^2 - 2}{x - 1}$?
   a $-2$           b $\frac{1}{6}$           c $\frac{1}{2}$
   d $2$           e $6$
**Answers to Practice Problems**

1a \(-4\sin x + 15x^{-4}\)  
1b \(-\sin x(1 + x^2) - 2x\cos x\)  
1c \(2\sqrt{3}x^{3-1} - \cot x \csc x + \frac{4}{3}x^3\)  
1d \(-\csc x + \csc x \sec^2 x\)  
1e \(\frac{45}{4}t^{-9/4} + \tan t \sec t\)

2a \(3\)  
2b \(\frac{38}{35}\)  
2c \(-1\)

4a \(x^4 \cot x + C\)  
4b \(\frac{2x\sqrt{x}}{\csc x} + C\)

5 \((a,b) = (-4, -3)\)

6a “19 weeks after September 1, a student studies 8 hours per week and is increasing his/her study time by 2 hours per week per week.”

6b “36 weeks after September 1, a student studies 1 hour per week and is decreasing his/her study time by 1 hour per week per week.”

7a \(y + 1 = -5(x - 3)\)

7b No.

7c \(-\frac{11}{25}\)

**Solutions to Practice Problems**

1a The derivative of \(\cos x\) is \(-\sin x\) and \(-\frac{5}{x^3}\) may be written as \(-5x^{-3}\). Then applying the Constant Multiple Rule to the first part and the Power Rule to the second (while realizing that the derivative of the constant \(\pi\) is zero), \(f'(x) = -4\sin x + 15x^{-4}\).

1b Let \(u = \cos x\) and \(v = 1 + x^2\); then \(u' = -\sin x\) and \(v' = 2x\), so applying the Quotient Rule to \(p = \frac{u}{v}\),

\[
\frac{dp}{dx} = \frac{-\sin x(1 + x^2) - 2x\cos x}{(1 + x^2)^2}
\]

You may simplify this if you are so inclined, but it’s unnecessary.

1c The derivative of the first term may be found using the Power Rule: \(\frac{d}{dx}(2x^{\sqrt{3}}) = 2\sqrt{3}x^{\sqrt{3}-1}\). The derivative of the second term is \(\frac{d}{dx}(\csc x) = -\cot x \csc x\), and the derivative of the last term is found by the Power Rule again:

\[
\frac{d}{dx}\left(\frac{x^3}{6}\right) = \frac{d}{dx}\left(\frac{1}{6}x^3\right) = \frac{4}{3}x^3 = \frac{4}{3}x^3.
\]

Adding the terms, \(b'(x) = 2\sqrt{3}x^{\sqrt{3}-1} - \cot x \csc x + \frac{4}{3}x^3\).

1d Let \(u = \tan x\) and \(v = \csc x\); then \(u' = \sec^2 x\) and \(v' = -\cot x \csc x\), so applying the Product Rule to \(y = uv\),

\[
\frac{dy}{dx} = \tan x(-\cot x \csc x) + \csc x \sec^2 x.
\]

Since \(\tan x \cot x = 1\), this is equivalent to \(\frac{dy}{dx} = -\csc x + \csc x \sec^2 x\).

1e Rewrite the first term as \(-9t^{-5/4}\) and apply the Power Rule to get \(\frac{d}{dt}\left(-9t^{-5/4}\right) = \frac{45}{4}t^{-9/4}\). The derivative of \(\sec t\) is \(\tan t \sec t\) and the derivative of \(-\frac{\sqrt{5}}{x}\) is zero since it’s constant, so add everything together to get \(\frac{dx}{dt} = \frac{45}{4}t^{-9/4} + \tan t \sec t\).

2a Expand \(\frac{d}{dx}(yz)\) with the Product Rule to get \(yz' + y'z\) and plug in the relevant values to get 3.

2b Expand \(\frac{d}{dx}\left(\frac{z}{y}\right) = \frac{yz' - y'z}{y^2}\) and plug in the relevant values to get \(\frac{38}{35}\).

3 Recognize this as \(\left.\frac{d}{dx}(x \cos x)\right|_{x=\pi}\) and note that \(\frac{d}{dx}(x \cos x) = -x \sin x + \cos x\). Plugging in \(x = \pi\) gives \(-1\).

4a Note that this looks like a Product Rule expansion, since \(\frac{d}{dx}(x^4) = 4x^3\) and \(\frac{d}{dx}(\cot x) = -\csc^2 x\). Thus an antiderivative is \(b(x) = x^4 \cot x + C\) for some constant C.
Note that this looks like a Quotient Rule expansion, and since the denominator has $\csc^2 x$, the denominator of the original function is $\csc x$. In the numerator we have $2x\sqrt{x}$ since its derivative is $3\sqrt{x}$, so an antiderivative is $h(x) = \frac{2x\sqrt{x}}{\csc x} + C$ for some constant $C$.

A horizontal tangent means that the tangent’s slope is zero. The derivative of the given function with respect to $x$ is $f'(x) = 3x^2 + 2ax + b$, and we know that $f(3) = -13$, so we can set up the system \[
\begin{align*}
3^2 + 3^2a + 3b + 5 &= -13 \\
3 \cdot 3^2 + 2(3)a + b &= 0
\end{align*}
\] which has the solution $(a,b) = (-4,-3)$.

We can read off the graph that $f'(3) = -5$ and are given that $f(3) = -1$, so we can write in point-slope form $y + 1 = -5(x - 3)$.

The given line has slope 4, but reading the graph gives $f'(-2) \approx -2.8 \neq 4$, so no.

The Quotient Rule gives $b'(x) = \frac{(x^2 + 1)f'(x) - 2xf(x)}{(x^2 + 1)^2}$, and the graph shows $f'(3) = -5$. We are given that $f(3) = -1$, so we plug in the necessary values in the Quotient Rule’s expression: \[
b'(3) = \frac{(3^2 + 1)(-5) - 2 \cdot 3(-1)}{(3^2 + 1)^2} = \frac{-11}{25}.
\]

The function has slope 4 on $(-3,-2)$, slope 0 on $(-2,1)$, and slope -1 on $(1,3)$, and is undefined elsewhere, so the graph of its derivative is as shown to the right.

Use the Quotient Rule with $u = x$ and $v = \tan x$. Therefore $u' = 1$ and $v' = \sec^2 x$, giving $f'(x) = \tan x - x \sec^2 x$. Now we can plug in $x = \frac{\pi}{4}$, which gives $f'(\frac{\pi}{4}) = 1 - \frac{\pi}{2}$, choice E.

Use the Quotient Rule, letting $f = uv$, so $f' = u'v + uv'$. The derivative of the quotient $\frac{f}{w}$ is thus $\frac{f'w - fw'}{w^2}$. We substitute what we know for $f$ and $f'$, giving $\frac{(u'v + uv')w - uvw'}{w^2} = \frac{u'vw + uv'w - uvw'}{w^2}$, which is equivalent to choice E.

Plug in $x = 2$ to get 2, choice D.