AP Calculus BC
Review — Chapter 12 (Sequences and Series), Part One

Things to Know and Be Able to Do

- Understand the distinction between sequences and series
- Understand the meaning of convergence (absolute and conditional), divergence, boundedness and how to test for each
- Understand arithmetic and geometric series and the partial sum formulas for each
- Understand the Integral Test and its applications along with the remainder estimate
- Understand *p*-series and know under what conditions they converge
- Know the comparison, ratio, and Alternating Series tests (root test optional)

Practice Problems

These problems should be done without a calculator. The original test, of course, required that you show relevant work for free-response problems.

1 Consider the sequence defined by $a_1 = 1$ and $a_{k+1} = \dfrac{a_k}{100}$ for $k \geq 1$.

a Write the first three terms of the sequence and an explicit formula for the n^{th} term of the sequence.

b If the sequence converges, find its limit. If the sequence diverges, explain why.

c Determine whether the series $\sum_{k=1}^{\infty} a_k$ converges or diverges. If it converges, find its sum.

2 Your calculator shows that $\displaystyle\sum_{k=1}^{n} \frac{1}{2k(k+1)} = \frac{1}{2} - \frac{1}{2(n+1)}$.

a Show this algebraically.

b Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k(k+1)}$ have a sum? How does the result of part **a** help to answer this question?

3 Determine whether each series converges or diverges. Justify your answers.

a $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{5}{n}\right)^n$ **b** $\displaystyle\sum_{n=1}^{\infty} \frac{2 + \sin n}{n^4}$ **c** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(n-1)!}$

4 Determine whether each series converges absolutely, converges conditionally, or diverges. Justify your answers.

a $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt[4]{n}}$ **b** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{e^n}$

5 It can be shown by the Alternating Series Test that the series $S = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3n}{10^n}$ converges.

a Give upper and lower bounds for the sum of this series.

b Find the smallest integer n such that S_n approximates S with an error no more than $\dfrac{1}{200}$. Justify your answer and explain whether this sum overestimates or underestimates the sum of the series.

6 Consider the series given by $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{5}{n^2}$.

a Use the Integral Test to show that this series converges. (Do not just make a statement about a *p*-series.)

b Find an upper bound on the error in using $S_k = \sum_{n=1}^{k} a_n$ to approximate $\sum_{n=1}^{\infty} a_n$. Hint: the error is given by $\sum_{n=k+1}^{\infty} a_n$.

c Based on part **b**, what value of *k* is needed so that S_k approximates the value of the series with an error of no more than 0.05?

7 What is the value of $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$?

a 1 **b** 2 **c** 4 **d** 6 **e** the series diverges

8 For what values of *p* does the infinite series $\sum_{n=1}^{\infty} \frac{n}{n^p + 1}$ converge?

a $p > 0$ **b** $p \geq 1$ **c** $p > 1$ **d** $p \geq 2$ **e** $p > 2$

9 Which of the following series diverge?

I $\sum_{n=0}^{\infty} \left(\frac{\sin 2}{\pi} \right)^n$ **II** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ **III** $\sum_{n=1}^{\infty} \frac{e^n}{e^n + 1}$

a III only **b** I and II only **c** I and III only **d** II and III only **e** I, II, and III

10 Which of the following sequences diverges?

a $\left\{ \frac{1}{n} \right\}$ **b** $\left\{ \frac{(-1)^{n+1}}{n} \right\}$ **c** $\left\{ \frac{2^n}{e^n} \right\}$ **d** $\left\{ \frac{n^2}{e^n} \right\}$ **e** $\left\{ \frac{n}{\ln n} \right\}$

11 If $s_n = \left(\frac{(5+n)^{100}}{5^{n+1}} \right) \left(\frac{5^n}{(4+n)^{100}} \right)$, to what number does the sequence $\{s_n\}$ converge?

a $\frac{1}{5}$ **b** 1 **c** $\frac{5}{4}$ **d** $\left(\frac{5}{4} \right)^{100}$ **e** the sequence does not converge

Answers

1a $a:\left\{1,\frac{1}{100},\frac{1}{10000},\frac{1}{1000000},\ldots,\frac{1}{100^{n-1}}\right\}$

1b The sequence converges to 0.

1c The series converges to $\frac{100}{99}$.

2a see solutions **2b** Yes.

3 The series in **a**, **b**, and **c** diverge, converge, and converge, respectively.

4a The series converges conditionally.

4b The series converges absolutely.

5a answers may vary, see solutions **5b** $n=3$

6a see solutions **6b** $\frac{5}{k}$ **6c** $k\geq 100$

7 c, **8** e, **9** d, **10** e, **11** a

Solutions

1a See **answers** section.

1b $\lim_{k\to\infty}\frac{1}{100^{k-1}}=\lim_{k\to\infty}\left(100\cdot\frac{1}{100^x}\right)=100\lim_{k\to\infty}\frac{1}{100^x}=100(0)=0.$

1c The series converges because it is a geometric series with constant factor $\frac{1}{100}$. Its sum is $\sum_{k=1}^{\infty}a_k=\frac{1}{1-\frac{1}{100}}=\frac{1}{\frac{99}{100}}=\frac{100}{99}$.

2a Applying the method of partial fractions to the summand gives $\frac{1}{2k(k+1)}=\frac{A}{2k}+\frac{B}{k+1}$, so $A(k+1)+B(2k)=1$, or $k(A+2B)+A=1$, giving the system $\begin{cases}A+2B=0\\A=1\end{cases}$ which solves to $(A,B)=(1,-\frac{1}{2})$. Thus $\sum_{k=1}^{n}\frac{1}{2k(k+1)}$

$$=\sum_{k=1}^{n}\left(\frac{1}{2k}+\frac{-\frac{1}{2}}{k+1}\right)=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{1}{2}\left(\left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right)+\cdots+\left(\cancel{\tfrac{1}{n}}-\frac{1}{n+1}\right)\right)=\frac{1}{2}\left(1-\frac{1}{n+1}\right)$$

$$=\frac{1}{2}-\frac{1}{2(n+1)}.$$

2b Yes, it does, because $\lim_{n\to\infty}\frac{1}{2(n+1)}=0$, so the only relevant part of the expression is the $\frac{1}{2}$. This is the series' sum.

3a Since the integral $\int_1^{\infty}\left(1-\frac{1}{x}\right)^x dx$ can be shown to diverge and $\left(1-\frac{5}{n}\right)^n\geq\left(1-\frac{1}{n}\right)^n\ \forall n\in[1,\infty)$, and the (divergent) integral must be less than the series, the series diverges.

3b Since $|\sin n|\leq 1\forall n$, we can treat this as $\sum_{n=1}^{\infty}\frac{3}{n^4}=3\sum_{n=1}^{\infty}\frac{1}{n^4}$. This converges because it is a p-series with $p=4>1$. Therefore the original series, too, converges.

3c The factorial indicates that we should use the ratio test. We therefore consider $\lim_{n\to\infty}\left|\frac{\frac{4^{n+1}}{n!}}{\frac{4^n}{(n-1)!}}\right|=\lim_{n\to\infty}\left|\frac{4^{n+1}}{n!}\cdot\frac{(n-1)!}{4^n}\right|$

$$=\lim_{n\to\infty}\left|\frac{4\cdot\cancel{4^n}\,\cancel{(n-1)!}}{\cancel{4^n}\,n\,\cancel{(n-1)!}}\right|=\lim_{n\to\infty}\left|\frac{4}{n}\right|=0.$$ Since $0<1$, the series is (absolutely) convergent.

4a We use the Alternating Series Test, considering $a_n=\frac{1}{\sqrt[4]{n}}=n^{-1/4}$. Clearly $(n+1)^{-1/4}\geq n^{-1/4}\forall n\in\mathbb{N}$ and $\lim_{n\to\infty}n^{-1/4}=0$. Therefore the series converges. However, without the alternation we simply have a p-series with $p=\frac{1}{4}<1$ which is divergent; therefore, the series is conditionally convergent.

4b We use the Alternating Series Test, considering $a_n = \frac{n^2}{e^n}$. Note that $n^2 \ll e^n$ for most *n* with which we are concerned; therefore, we instead consider the rather simpler $a_n = \frac{1}{e^n} = e^{-n}$. This is a convergent geometric series. Since taking $\frac{1}{e^n} > \frac{n^2}{e^n}$ as the summand also converges, this series is absolutely convergent.

5a The series meets the conditions of the Alternating Series Test. Therefore consider two consecutive terms of the sequence of partial sums—say, those with $n=5$ and $n=6$. These are approximately 0.24795 and 0.24793. The sum of the series is between these two values.

5b The first terms of the series are as follows: $\frac{3}{10} - \frac{3}{50} + \frac{9}{1000} - \frac{3}{2500} + \frac{3}{20000} \cdots$. Since $\frac{3}{2500} < \frac{1}{200}$ and the series meets the conditions of the Alternating Series Test, $n=3$.

6a Let $f(n) = a_n = \frac{5}{n^2}$. $f(n)$ is positive, continuous, and decreasing for $n \in [1, \infty)$. Consider $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{5}{x^2}dx$ $= \lim_{a\to\infty}\left(-\frac{5}{x}\right)\Big]_{x=1}^{a} = 5$. Since the integral converges, the series must converge.

6b The error must be less than the error found by using an integral to approximate the sum. Consider $\int_1^k 5x^{-2}dx$ $+\int_k^\infty 5x^{-2}dx$, where the first term represents an approximation to the series' sum and the second represents an approximation to the error. The second term evaluates to $\frac{5}{k}$.

6c We are interested in $\frac{5}{k} \le 0.05 \Rightarrow k \ge 100$.

7 Rewrite this as $2\sum_{n=1}^{\infty}\frac{2^n}{3^n} = 2\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$. The series is geometric and sums to 2, so the total is $2(2) = 4$, choice **c**.

8 The $+1$ in the denominator is insignificant, so consider $\sum_{n=1}^{\infty}\frac{n}{n^p} = \sum_{n=1}^{\infty}\frac{1}{n^{p-1}}$. A *p*-series converges for $p > 1$ but here we are dealing with a "$p-1$-series". Therefore the series converges for $p > 2$, which is choice **e**.

9 Series I is a geometric series with $r = \frac{\sin 2}{\pi} \approx 0.289 < 1$ so it does not diverge. Series II is a *p*-series with $p = -\frac{1}{3} \le 1$ so it diverges. Series III is extremely difficult to work with, but given that the correct answer choice must include II and not I, only choice **d** is possible.

10 Choices **a** and **b** both converge to 0. Choice **c** is geometric with $r = \frac{2}{e} \approx 0.736 < 1$ so it converges. Choice **d** was dealt with in problem **4b**, where it was shown to converge. For choice **e**, note that *n* grows faster than $\ln n$, so the sequence must diverge. Choice **e** is thus correct.

11 $\left(\frac{(5+n)^{100}}{5^{n+1}}\right)\left(\frac{5^n}{(4+n)^{100}}\right) = \frac{(5+n)^{100}\,\cancel{5^n}}{5\cdot\cancel{5^n}\,(4+n)^{100}} = \frac{1}{5}\left(\frac{5+n}{4+n}\right)^{100}$. Since $\lim_{n\to\infty}\left(\frac{5+x}{4+x}\right)^{100} = 1$, we have $s_n = \frac{1}{5}(1) = \frac{1}{5}$, or **a**.