AP Calculus BC
Review — Chapter 10 (Differential Equations)

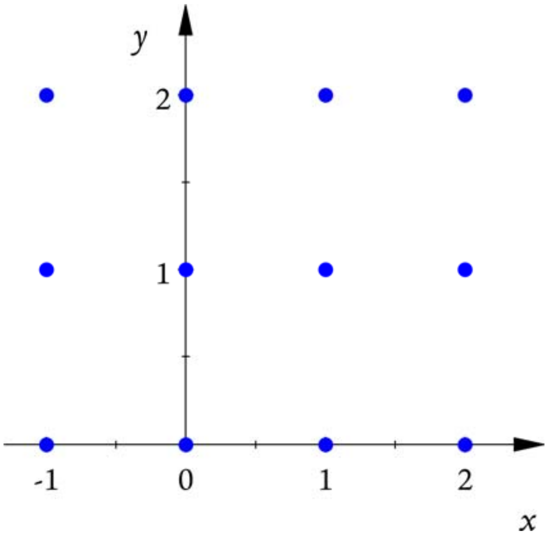
Things to Know and Be Able to Do

- Interpret first-order differential equations (that is, those involving only a function and its first derivative)
- Show that a particular function satisfies a differential equation of any order
- Solve first-order separable differential equations
- Sketch slope fields for first-order differential equations
- Approximate solutions to first-order differential equations with Euler's method
- Use first-order differential equations to model population growth and decay for various types of "populations", and interpret these equations without necessarily explicitly solving them
- Know the properties of exponential and logistic growth curves

Practice Problems

These problems may be done with a calculator except where noted otherwise. The original test, of course, required that you show relevant work.

1 Consider the differential equation $\frac{dy}{dx} = 2x - y$.



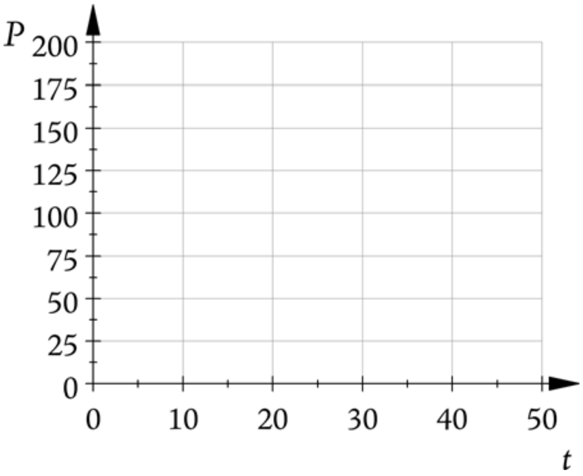
a On the axes provided at right, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0,1)$.

b The solution curve that passes through the point $(0,1)$ has a local minimum at $x = \ln\frac{3}{2}$. Find the y-coordinate of this local minimum.

c Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

d Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part **c** is less than or greater than $f(-0.4)$. Explain your reasoning.

2 The size of a certain animal population is governed by the equation $1000\frac{dP}{dt} = P(100 - P)$, where $P(t)$ is the number of individuals in the colony at time t. The initial population is known to be 200 individuals.



a If you use your calculator to solve for $P(t)$ in terms of t, it will give $P = \frac{200e^{t/10}}{2e^{t/10} - 1}$. Show how to obtain this solution without using your calculator.

b On the axes provided at right, sketch a graph of $P(t)$ as a function of t on the axes below for $0 \le t \le 50$ and $0 \le P \le 200$.

c Show use of calculus to find $\lim_{t \to \infty} P(t)$.

3 Let $R(t)$ be the number of trigonometry facts that a student remembers at time t, where t is measured in weeks. Write a differential equation using the function $R(t)$ that models each situation described. *The situations are independent of each other. You may introduce additional symbols if you define them.*

a While taking precalculus, a student learns and remembers new trigonometry facts at a rate of 7 per week.

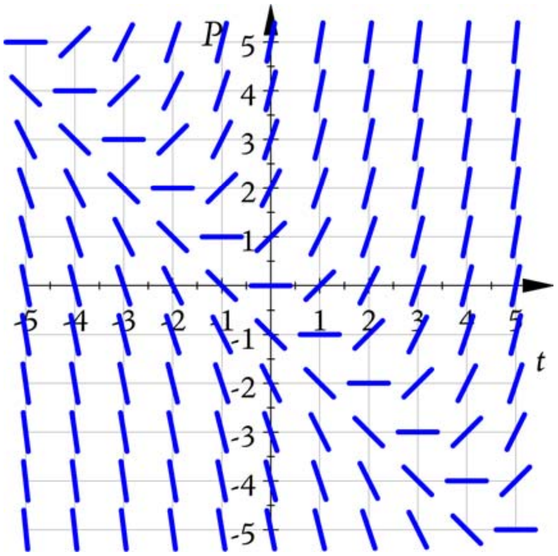
b Each week, while taking summer school geometry, students forget 10% of the trigonometry facts that they previously knew.

c Each week, an algebra student learns new trigonometry facts at a rate directly proportional to the number of facts he has already learned and the difference between the number of facts he has learned and 90% of the total number that there is to learn.

4 Consider the differential equation $\frac{dy}{dx}=\frac{3-x}{y}$.

a Let $y=f(x)$ be the particular solution to the given differential equation for $1<x<5$ such that the line $y=-2$ is tangent to the graph of *f*. Find the *x*-coordinate of the point of tangency, and determine whether *f* has a local maximum, local minimum, or neither at this point. Justify your answer.

b Let $y=g(x)$ be the particular solution to the given differential equation for $-2<x<8$, with the initial condition $g(6)=-4$. Show non-calculator work to find $y=g(x)$.



5 Shown at right is the slope field for which of the following differential equations?

a $\frac{dy}{dx}=1+x$ **b** $\frac{dy}{dx}=x^2$ **c** $\frac{dy}{dx}=x+y$

d $\frac{dy}{dx}=\frac{x}{y}$ **e** $\frac{dy}{dx}=\ln y$

6 If $\frac{dy}{dx}=(1+\ln x)y$ and if $y=1$ when $x=1$, then $y=$

a $e^{(x^2-1)/x^2}$ **b** $1+\ln x$ **c** $\ln x$ **d** $e^{2x+x\ln x-2}$ **e** $e^{x\ln x}$

7 Let $y=f(x)$ be the solution to the differential equation $\frac{dy}{dx}=x+y$ with the initial condition $f(1)=2$. What is the approximation for $f(2)$ if Euler's method is used starting at $x=1$ with a step size of 0.5?

a 3 **b** 5 **c** 6 **d** 10 **e** 12

8 The number of moose in a national park is modeled by the function *M* that satisfies the logistic differential equation $\frac{dM}{dt}=0.6M\left(1-\frac{M}{200}\right)$, where *t* is the time in years and $M(0)=50$. What is $\lim_{t\to\infty}M(t)$?

a 50 **b** 200 **c** 500 **d** 1000 **e** 2000

9 Population *y* grows according to the equation $\frac{dy}{dt}=ky$, where *k* is a constant and *t* is measured in years. If the population doubles every 10 years, then the value of *k* is most nearly

a 0.069 **b** 0.200 **c** 0.301 **d** 3.322 **e** 5,000

10 A function satisfies the differential equation $\frac{dy}{dt}=y^4-13y^3+30y^2$. For what values of *y* is $y(t)$ decreasing?

a $(-\infty,3)\cup(10,\infty)$

b $(-\infty,0)\cup(0,3)\cup(10,\infty)$

c $(-\infty,0)\cup(10,\infty)$

d $(0,3)\cup(10,\infty)$

e $(3,10)$

Answers

1a see solutions

1b $2\ln\frac{3}{2}$ **1c** 1.52

1d $2-2x+y$; less than

2a, 2b see solutions

2c 100

3a $\frac{dR}{dt}=7$ **3b** $\frac{dR}{dt}=-0.1R$

3c $\frac{dR}{dt}=kR(R-0.9T)$ for constant k and T the total number of facts

4a $x=3$; local minimum

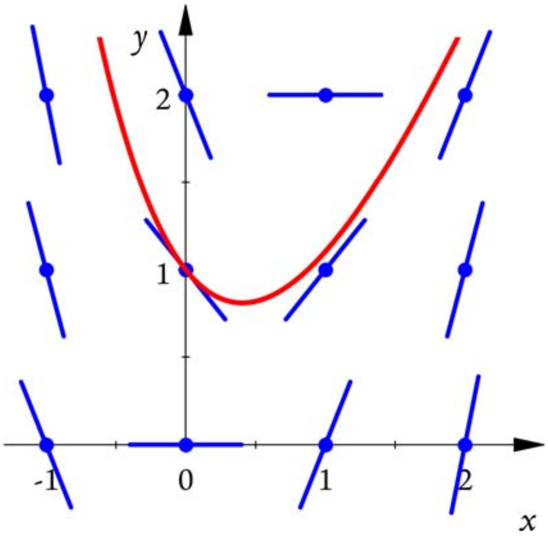
4b $y=-\sqrt{6x-x^2+16}$

5 c **6** e **7** c

8 b **9** a **10** e

Solutions

1a At each of the points shown, find the slope using $\frac{dy}{dx}=2x-y$ and the coordinates of the point. Then begin at the point $(0,1)$ and follow approximately the slopes shown to get the solution curve through that point; your result will likely not look quite as precise as mine (shown at right), but the general shape should be the same.



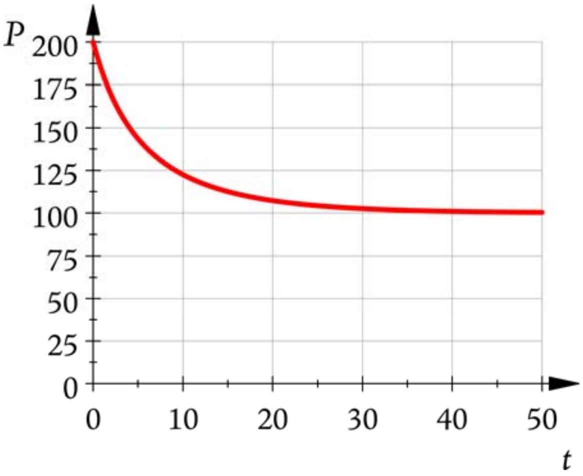
1b The fact that this point is a local minimum means that $\frac{dy}{dx}=0$ at that point. Therefore $0=2x-y$, so $2x=y$. Given that $x=\ln\frac{3}{2}$, we know that $y=2\ln\frac{3}{2}$.

1c The "two steps of equal size" means that we approximate $f(-0.2)$, then use that to find the approximation to $f(-0.4)$. Euler's method is applied to this differential equation as follows: $f(-0.2)\approx f(0)+(-0.2)f'(0)=1+(-0.2)(-1)=1.2$. Then $f(-0.4)\approx f(-0.2)+(-0.2)f'(-0.2)=1.2+(-0.2)(-1.6)=1.52$.

1d Differentiating both sides of $\frac{dy}{dx}=2x-y$ with respect to x gives $\frac{d^2y}{dx^2}=2-\frac{dy}{dx}$, into which we substitute the known $\frac{dy}{dx}$ to get $\frac{d^2y}{dx^2}=2-(2x-y)=2-2x+y$. At and around $x=-0.4$, $\frac{d^2y}{dx^2}$ is positive, which means the solution curves in that vicinity are concave up. Therefore the approximation of 1.52 is less than the actual value.

2a The equation separates to $\frac{dP}{P(100-P)}=\frac{dt}{1000}$, so integration with partial fractions will make the left-side integrand take the form $\frac{A}{P}+\frac{B}{100-P}=\frac{1}{P(100-P)}$. The denominators can be cleared to give $A(100-P)+BP=1$, which we convert into the system $\begin{cases}B-A=0\\100A=1\end{cases}$ which solves to $(A,B)=\left(\frac{1}{100},\frac{1}{100}\right)$. Therefore we have $\int\left(\frac{1/100}{P}+\frac{1/100}{100-P}\right)dP=\int\frac{dt}{1000}$, or $\frac{1}{100}\ln P-\frac{1}{100}\ln(100-P)=\frac{1}{1000}t+C$. Logarithm rules allow this to be rewritten as $\frac{1}{100}\ln\frac{P}{100-P}=\frac{1}{1000}t+C$, or $\ln\frac{P}{100-P}=\frac{1}{10}t+C$ for a different C. Exponentiating both sides gives $\frac{P}{100-P}=Ce^{t/10}$ which we rearrange to find $P=100Ce^{t/10}-PCe^{t/10}$, and again for $P\left(1+Ce^{t/10}\right)=100Ce^{t/10}$. Finally, we arrive at $P=\frac{100Ce^{t/10}}{Ce^{t/10}+1}$. Since we know that $P(0)=200$, we plug in $200=\frac{100Ce^{0}}{Ce^{0}+1}$, which gives $C=-2$. Therefore $P(t)=\frac{100(-2)e^{t/10}}{-2e^{t/10}+1}=\frac{\cancel{(-1)}200e^{t/10}}{\cancel{(-1)}\left(2e^{t/10}-1\right)}=\frac{200e^{t/10}}{2e^{t/10}-1}$, which is what we wanted $\left(W^{5}\right)$.

2b This might be easier to do after having completed part **c**, so that one knows the equilibrium population. The graph is shown at right.



2c $P(t)=\frac{200e^{t/10}}{2e^{t/10}-1}$ is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply to it l'Hôpital's rule. Differentiating the numerator and denominator gives $\frac{20\cancel{e^{t/10}}}{\frac{1}{5}\cancel{e^{t/10}}}$. Therefore the limit is $\frac{20}{1/5}=100$.

3 If you are unable to understand these answers, please see your teacher for explanation.

4a The slope of the tangent line at the point in question is clearly 0, so we solve $0=\frac{3-x}{y}$ to get $x=3$ by the Zero Product Property. Now we find $\frac{d^2y}{dx^2}=\frac{-y-\frac{dy}{dx}(3-x)}{y^2}$ by Quotient Rule-aided implicit differentiation. Knowing that the point $(3,-2)$ is on the graph and the derivative at that point is 0 lets us find $\frac{d^2y}{dx^2}=\frac{-(-2)-(0)(3-3)}{(-2)^2}$ $=\frac{2}{4}=\frac{1}{2}$. This is positive, so *f* has a local minimum at that point.

4b We separate $\frac{dy}{dx}=\frac{3-x}{y}$ into $y\,dy=(3-x)dx$. Integrating both sides gives $\frac{1}{2}y^2=3x-\frac{1}{2}x^2+C$, into which we can plug $(x,y)=(6,-4)$ to get $8=18-18+C$, so $C=8$. Therefore $y^2=6x-x^2+16$, meaning $y=\pm\sqrt{6x-x^2+16}$. To determine whether we are interested in the positive or negative square root, we plug in the point $(6,-4)$ again. The positive square root gives $-4=4$, which is not true for most values of 4, while the negative square root gives $-4=-4$. This is true, so $y=-\sqrt{6x-x^2+16}$.

5 Notice that $\frac{dy}{dx}=-1$ everywhere along the line $y=-x-1$. Plugging in $-x-1$ for y and -1 for $\frac{dy}{dx}$ in each of the choices gives the following results:

a $-1=1+x$
b $-1=x^2$
c $-1=x+(-x-1)$, or $-1=-1$
d $-1=\frac{x}{-x-1}$
e $-1=\ln(-x-1)$

The equation for option **c** is a true statement; the others, not so much (in the case of **b** and **d**, they need not even have solutions for real *x*). Therefore **c** is correct.

6 Separate the equation into $\frac{dy}{y}=(1+\ln x)dx$ and integrate each side to get $\ln y = x\ln x + C$. Exponentiating each side gives $y=Ce^{x\ln x}$; given the initial condition, $y=e^{x\ln x}$, choice **e**.

7 Given $f(1)=2$, we approximate $f(1.5)\approx f(1)+0.5f'(1)=2+0.5(3)=3.5$. Then $f(2)$ is approximated by $f(2)\approx f(1.5)+0.5f'(1.5)=3.5+0.5(4.7)=6$, choice **c**.

8 If the size of a logistic population M at time t is modeled by an equation of the form $\frac{dM}{dt}=kM\left(1-\frac{M}{P_{\max}}\right)$, $P_{\max}$ is the maximum population. In this case, it is 200, choice **b**.

9 Solve the equation as follows by separation: $\int\frac{dy}{y}=\int k\,dt$ so $\ln|y|=kt+C$, or $y=Ce^{kt}$. Then we can set up the following system based on knowledge of the doubling time: $\begin{cases} y(t)=Ce^{kt} \\ 2y(t)=Ce^{k(t+10)} \end{cases}$. Therefore $2\cancel{C}e^{kt}=\cancel{C}e^{k(t+10)}$, or $2\cancel{e^{kt}}=\cancel{e^{kt}}e^{10k}$, so $k=\frac{1}{10}\ln 2\approx 0.069$, choice **a**.

10 The fact that $y(t)$ is decreasing means that $\frac{dy}{dt}$ is negative. The graph of $\frac{dy}{dt}$ as a function of t (which you can reproduce on your calculator by graphing $y_1=x^4-13x^3+30x^2$, treating t as x and $\frac{dy}{dt}$ as y) is shown at right; $\frac{dy}{dt}$ is negative on $(3,10)$ only, choice **e**.