AP Calculus BC
Lesson 11.2 Parametric and Vector Calculus

1. By the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

   a) Solve the chain rule equation above for $\frac{dy}{dx}$

   b) Use your result from part (a) to find the slope of the tangent line to the parametric graph, $x(t) = t^2 + 1$, $y(t) = 3t^4$ at $t = 2$.

   c) Write the parametric equation in rectangular form and find the slope of the tangent line at $x = 5$.

   d) For what values of t is the tangent line to the parametric graph a vertical line? A horizontal line?

2. Given $\mathbf{R}(t) = (3t)\mathbf{i} + (1 - 9t^2)\mathbf{j} = \langle 3t, 1 - 9t^2 \rangle$,

   a) find $\frac{dy}{dx}$ in terms of t.

   b) find $\frac{d^2y}{dx^2}$ in terms of t.

   c) Write the function in Cartesian form. Check your answers to a) and b) against the Cartesian form of the first and second derivatives.

3. Given $\mathbf{R}(t) = \langle 4\cos t, 4\sin t \rangle$ for $0 \le t \le 2\pi$,

a) sketch the graph of **R**(*t*).

b) find $\dfrac{dy}{dt}$ at $t = \frac{\pi}{3}$. What does this mean?

c) find $\dfrac{dx}{dt}$ at $t = \frac{\pi}{3}$. What does this mean?

d) find the slope of the tangent line to the graph at the point where $t = \frac{\pi}{3}$.

e) find $\dfrac{d^2y}{dt^2}$ at $t = \frac{\pi}{3}$. What does this mean?

f) find $\dfrac{d^2x}{dt^2}$ at $t = \frac{\pi}{3}$. What does this mean?

g) find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$ in terms of *t* and evaluate at $t = \frac{\pi}{3}$. What does this mean?

h) find $\dfrac{d^2y}{dx^2}$ in terms of *t* and evaluate at $t = \frac{\pi}{3}$. What does this mean?

i) find the velocity vector **v**(t), and find $\mathbf{v}\left(\frac{\pi}{3}\right)$. Draw this vector on the graph.

j) find the acceleration vector **a**(*t*), and $\mathbf{a}\left(\frac{\pi}{3}\right)$. Draw this vector on the graph.

4. Consider **R** = (2 + cos(t))**i** – (3 – sin(t))**j.**

a) Find a Cartesian graph for this curve.

b) Find parametric equations for x and y.

c) Find the velocity and acceleration vectors.

d) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

5. Given $\begin{cases} x(t) = t^2 + t \\ y(t) = t^2 - t \end{cases}$

(a) Find all places where the tangent line to the curve is horizontal.

(b) Find all places where the tangent line to the curve is vertical.

6. Given the vector-valued function **R**(t) = ln(t + 1)**i** + $\tan^{-1}(t)$**j,** find the velocity and acceleration vectors.

7. A projectile with initial velocity of 2500 feet per second is fired at an angle of 32 degrees as measured from the horizontal.

   (a) Describe, using parametric equations or a vector function, the motion of the projectile, letting the parameter *t* represent time.

   (b) What is the maximum height the projectile attains?

   (c) What is the range of the projectile? (How far, horizontally, does the projectile travel before hitting the ground?)

8. Draw a complete graph of
$$x(t) = 9\cos(t)$$
$$y(t) = 4\sin(t)$$

   a) What is the rectangular equation of this parametric curve?

   b) What is the rectangular equation of the tangent line to this curve at the point on the curve where $t = \pi/4$?

9. A particle moves according to the vector function $\mathbf{R}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j}$,

a) Find the velocity and acceleration vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

b) Find an expression for the speed of the particle at any time t.

c) Find the total distance traveled from $t = 0$ to $t = \pi$.

d) Write the function in parametric form.

e) Find the arc length of the curve from $t = 0$ to $t = \pi$.

10. Given $\mathbf{R}(t) = \sqrt{2t+1}\,\mathbf{i} + (t-1)^2\mathbf{j}$.

(a) Find the velocity and acceleration vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Find an expression that gives the speed of the particle at any time t.

11. Consider the parametric curve $\begin{cases} x(t) = \ln(\sin(t)) \\ y(t) = t + 1 \end{cases}$ from $t = \dfrac{\pi}{6}$ to $t = \dfrac{\pi}{2}$.

Find the length of the parametric curve.

12. [1994-BC3] A particle moves along the graph $y = \cos(x)$ so that the x-component of the acceleration is always 2. At time $t = 0$, the particle is at the point $(\pi, -1)$ and the velocity vector of the particle is (0, 0).

(a) Find the x- and y- coordinates of the position of the particle in terms of t.

(b) Find the speed of the particle when its position is (4,cos(4)).

13. [1982 BC6] Point $P(x,y)$ moves in the xy-plane in such a way that $\dfrac{dx}{dt} = \dfrac{1}{t+1}$ and $\dfrac{dy}{dt} = 2t$ for $t \geq 0$.

(a) Find the coordinates of P in terms of t if, when $t = 1$, $x = \ln(2)$ and $y = 0$.

(b) Write an expression for y in terms of x.

(c) Find the average rate of change of <u>*y with respect to x*</u> as t varies from 0 to 4.

(d) Find the instantaneous rate of change of y with respect to x when $t = 1$.

14. Use integration with t to find the area bounded by the curve $x = \cos t$, $y = e^t$ for $0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$. *Hint: think about* $\int y\ dx$ *in terms of t.*

15. Find the surface area when the curve $x = \ln\left(\sin\left(t\right)\right)$ and $y = t + 1$ for $\frac{\pi}{6} \le t \le \frac{\pi}{2}$ is revolved around the y-axis.

16. [1978 BC7] A particle moves in the plane so that at any time t, $0 \le t \le 1$, its position is given by $x = \frac{1}{4}e^{8t} - 2t$ and $y = e^{4t}$. Let C denote the path traveled by the particle.

(a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

(b) Find the arc length C.

(c) Set up [and evaluate] an integral, involving only the variable t, that represents the area of the surface generated by rotating C about the y-axis.

17. Consider **R**(t) = (2 + cos(t))**i** - (3 – sin(t))**j**,

Q(t) = (e^t - 1)**i** + (e^t + 1)**j** and

S(t) = $\left(\dfrac{t^2-2t-3}{t-3}\right)$**i** + $\left(\dfrac{t^2-5t+6}{t-3}\right)$**j.**

a) Find $\lim\limits_{t\to 3}$ **S**(t).

b) Find the acceleration and velocity vectors for the function **R**(t)

c) Demonstrate that the speed of **R**(t) is constant.

d) Find $\dfrac{d}{dt}\left(\left|Q'(t)\right|\right)$

e) Find $\dfrac{d}{dt}\left[R(t)+Q(t)\right]$.

f) $\dfrac{d}{dt}\left[R(t)\bullet Q(t)\right]$.

g) Determine $\int Q(t)dt$.

h) Determine an expression for the cosine of the angle θ between **R**(t) and **Q**(t).

18. A particle moves in the plane with constant acceleration vector **a** = a**j**. Show that its path is a parabola or a straight line.

19. The world's record for popping a champagne cork is 109 feet, 6 inches, held by Captain Michael Hill of the British Royal Artillery. Assuming that Captain Hill held the neck of the bottle at a 45° angle, how fast was the cork going as it left the bottle?

20. The Civil War Mortar Dictator weighed so much (17,120 lb) that it had to be mounted on a railroad car. It had a 13 inch bore and fired a 200 lb shell with a 20 lb powder charge. It was made by Mr. Charles Knapp in the Pittsburgh Ironworks. It was used by the Union Army in 1864 in the siege of Petersburg, VA. How far did it actually shoot? Different records claimed 4325 yards and 4752 yards. Assuming a 45° firing angle, what muzzle speeds do these require?